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An Equation of Motion with Quantum Effect in Spacetime *

Jyh-Yang Wu†
Department of Mathematics , National Chung Cheng University,
Ming-Hsiung, Chia-Yi 621, Taiwan

Abstract

In this paper, we shall present a new equation of motion with Quantum effect in spacetime. To do so, we propose a classical-quantum duality. We also generalize the Schordinger equation to the spacetime and obtain a relativistic wave equation. This will lead a generalization of Einstein's formula $E = m_0c^2$ in the spacetime. In general, we have $E = m_0c^2 + \frac{\hbar^2}{12m_0}R$ in a spacetime.

1 Introduction

The effort to combine the theories of General Relativity and Quantum Mechanics can be tracked to the very beginning of these two theories. Since the middle of this century, the study of Quantum gravity has attracted a lot of attention. The usual way to combine these two theories is to start from Quantum Mechanics (QM), via Special Relativity (SR), and then reach General Relativity (QR). This gives

Physical Approach (I) :
Quantum Mechanics $\mapsto$ Special Relativity $\mapsto$ General Relativity.

Following this path, one can obtain the celebrated Dirac equation and the Klein-Gordon equation. However, the advance to the level of General Relativity along this line has not yet succeeded. In other words, one now has more knowledge about the quantum effect in Special Relativity, but is not able to combine the quantum effect with the general relativity effect, i.e., the quantum effect in spacetime.

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†email:jywu@math.ccu.edu.tw

To unify the theories of Quantum Mechanics and General Relativity one need to take care of three major effects : quantum effect, (special) relativity effect and gravitation effect. The gravitation effect is also called the general relativity effect which is an effect of the curvedness of the spacetime. This physical approach from Quantum Mechanics, via Special Relativity, to General Relativity tries to catch the quantum-relativity effect first and then try to combine it with the gravitation effect. In this note we try to propose another way from Quantum Mechanics to General Relativity, via Riemannian Geometry (RG). Namely, we introduce

Geometric Approach (II) :
Quantum Mechanics $\mapsto$ Riemannian Geometry $\mapsto$ General Relativity.

This approach tries to catch the quantum-gravitation effect first and then combine it with the relativity effect. Thus, the first approach is more physical, while the second more geometric. A common fundamental problem in both physics and geometry is to deal with the motion of objects in a suitable underlying space. This is why there are so many interactions between these two fields.

In order to illustrate the importance of the Geometric Approach (II), we shall point out, in this note, a new equation of motion with quantum effect in spacetime. To demonstrate why the geometry plays an important role in physical world, we would like to point out that both theories of Quantum Mechanics and General Relativity are, in some respects, geometric. On one hand, it is well-known that in Einstein's theory of General Relativity the gravitation is viewed as the curved effect of spacetime. In other words, General Relativity uses the curved spacetime to unify space, time and gravity. Thus, General Relativity is geometric. On the other hand, Feynman's path integral formulation of the theory of Quantum Mechanics provides us a sum-over-all-path thinking. This amounts to the consideration of all possible trajectories of a particle. Hence this viewpoint is also geometric. These Observations leads us to realize the important role of geometry that many possibly play in these two theories and also give us a sufficient hint to put the theory of Riemannian Geometry as the intermediate step from Quantum Mechanics to Geometry Relativity even through at the first sight, the theory of Riemannian Geometry may seem a little bit exotic from the viewpoint of physics.

In section two we shall introduce a generalized Newton's equation of motion. For a massive particle with rest mass m_o, the timelike trajectory is governed by the generalized Newton's equation of motion:

$$F = m_o \nabla_{\gamma'(\tau)} \gamma'(\tau) - \frac{\hbar^2}{12 m_o} \nabla R \tag{1.1}$$

where ∇R is the gradient of the scalar curvature R as given by (2.2), h the Planck constant and F the external force. For a massless particle, like photons, the trajectory is given by the equation :

$$\nabla_{\gamma'(\tau)}\gamma^{'}(\tau) = \frac{\lambda^2 c^2}{48\pi^2}\nabla R \tag{1.2}$$

without external force where λ stands for the wavelength of the massless particle.

The Equation (1.2) does not contain the quantum parameter $\hbar$. However, it comes from the consideration of quantum effect on curved space-time. Moreover, this equation will give us a new phenomenon of "gravitational rainbow" and also provide a new explantation of the evaporation of a black hole rather than that given by Hawking. See [Wu3-4] for detailed discussion of these new effects.

In section three we give a detailed explanation how this equation arises naturally. In section four we shall also develop a new relativistic wave equation:

$$i\hbar\frac{\partial\Phi(x,\tau)}{\partial\tau} = -\frac{\hbar^2}{2m_o}\Box\Phi(x,\tau) + V(x)\Phi(x,\tau) - \frac{m_o c^2}{2}\Phi(x,\tau). \tag{1.3}$$

This is a natural generalization of Schrödinger equation in spacetime since a steady-stay of this equation gives the Klein-Gordon equation. As a byproduct, a new action S_{GR} with the general relativity effect is also discussed in (4.6) and as a special case the formula $E = mc^2$ comes out naturally.

2 The equation of motion with quantum effect

In this section we shall formulate Einstein's General Relativity and propose our equation of motion with quantum effect in spacetime.

Following the Geometric Approach (II) to investigate the quantum effect in General Relativity, we shall employ the geometric viewpoint to study these two theories. First we give a formulation of Einstein's General Relativity which consists of two components :

(GR-I) Geometric Component. *The space is determined by Einstein's Field Equation ([We]) :*

$$R_{ij} - \frac{R}{2}g_{ij} = -\frac{8\pi G}{c^4}T_{ij} \tag{2.1}$$

where R_{ij} denotes the Ricci curvature of the Lorentz metric g_{ij}, R the scalar curvature, c the speed of light and T_{ij} the energy-momentum

tensor.

Taking the trace of both sides of the equation (2.1), one obtains the scalar curvature

$$R = \sum_i \frac{8\pi G}{c^4} T_{ij}. \tag{2.2}$$

This scalar curvature will play an important role in our formulation of equation of motion from the view point Mach's principle. This component (GR-I) describes how the geometry of the spacetime forms. Since Einstein gave his description of the general relativity, a lot of variants have tried to add some new variables or freedom to modify Einstein's theory, like Brans-Dicke scalar-tensor theory. However, it turns out so far that Einstein's theory is the most elegant and simple one that has passed several tests.

(GR-II) Physical Component. *The equation of motion for a photon or massive particle with mass m_o moving in the spacetime is given by the geodesics equation :*

$$\nabla_{\gamma'(\tau)} \gamma^{'}(\tau) = 0 \tag{2.3}$$

where ∇ denotes the covariant derivative associated to the Lorentz metric tensor g_{ij} for the spacetime and $\gamma(\tau)$ is the timelike trajectory of the particle under consideration.

This Component tells us how a massless or massive particle moves in spacetime. The equation of motion with quantum effect that we would like to propose to replace (GR-II) is following :

(GR-II') Physical Component with Quantum Effect. *For a massless particle, like photon, the trajectory is given by the equation :*

$$\nabla_{\gamma'(\tau)} \gamma^{'}(\tau) = \frac{\lambda^2 c^2}{48\pi^2} \nabla R \tag{2.4}$$

without external force where λ stands for the wavelength of the massless particle. While for massive particle with rest mass m_o, the timelike trajectory is given by the generalized Newton equation of motion :

$$F = \nabla_{\gamma'(\tau)} \gamma^{'}(\tau) - \frac{\hbar^2}{12 m_o} \nabla R \tag{2.5}$$

where ∇R is the gradient of the scalar curvature R, $\hbar$ the Planck constant and F the external force. Note that the term $\frac{\hbar^2}{12 m_o} \nabla R$ is the

quantum-gravitation effect in spacetime. It is also worth noticing that vector $\nabla R(x)$ should be timelike for any point x in spacetime. This can be seen easily from the fact that if we place a rest particle with mass m_o at x, then it should be accelerated in the direction of $\nabla R(x)$, and hence the vector $\nabla R(x)$ is timelike.

Before we explain how this term comes out, we discuss this term from the viewpoint of Mech's principle. It is well-known that Mech's principle was considered to be of great importance and value to Einstein who tried to incorporate it into his general theory of relativity. However, Einstein's formulation of his general relativity (GR-I) (GR-II) is only semi-Machian. Since (GR-I) indicates that the energy-momentum tensor T_{ij} determines the geometry of spacetime, (GR-I) is distribution can put on the motion of particles, so it is not Machian. In our formulation (GR-II') we also include this effect into consideration since the term $\frac{\hbar^2}{12m_o}\nabla R$ is related to the energy-momentum tensor T_{ij} as given in (2.2). Therefore, these two components (GR-I) and (GR-II) together give a full Machian theory of General Relativity.

3 The quantum-gravitation effect : $\frac{\hbar^2}{12m_o}\nabla R$ and $\frac{\lambda^2 c^2}{48\pi^2}\nabla R$

In this section we shall illustrate theoretically how to obtain the term $\frac{\hbar^2}{12m_o}\nabla R$ in (2.5) and why we believe that the generalized Newton's equation of motion (2.5) in spacetime should be the one we search for when we take the quantum-gravitation effect into consideration. To do so, we recall the observations of Dirac and Feynman about the path integral formulation of Quantum Mechanics in the 3-space $\mathbb{R}^3$.

Dirac observed in [Di] the Lagrangian or the action also played an important role in Quantum Mechanics, especially from the viewpoint of transformation theory. Feynman pushed further to obtain his path integral formulation. To be more specific, one considers the Schrödinger equation :

$$i\hbar\frac{\partial\Phi(x,t)}{\partial t} = -\frac{\hbar^2}{2m_o}\nabla^2\Phi(x,t) + V(x)\Phi(x,t) \tag{3.1}$$

for a particle with mass m_o in $\mathbb{R}^3$. Now the fundamental solution $K(b,a)$ can be written as a path integral :

$$K(b,a) = \int_a^b e^{\frac{i}{\hbar}S[\gamma(t)]}\mathcal{D}\gamma(t)$$

where $S[\gamma(t)]$ denotes the action along the path $\gamma(t)$ from a to b; a denotes the particle at the position x_0 and time t_0, and b at the position x_1 at time t_1. In brief, $a = (x_0, t_0)$ and $b = (x_1, t_1)$.

The key points that concern us are following :

(1) The equation of motion can be obtained from the Euler-Lagrangian equation of the action S;

(2) The action S is inscribed in the short-time amplitude of the path integral formulation for the fundamental solution of the Schrödinger equation.

These together give us a useful duality ([Di] [f]) :

Classical-Quantum Duality. *The Newton's equation of motion is called in the short-time amplitude of the path integral as in the diagram :*

$$F = m_o\gamma^{''}(t) \Longleftrightarrow K(b,a) = \int_a^b e^{\frac{i}{\hbar}S[\gamma(t)]}\mathcal{D}\gamma(t).$$

where $S[\gamma(t)] = \int_{t_0}^{t_1} \frac{m_o}{2} < \gamma^{'}(t), \gamma^{'}(t) > -V(\gamma(t))dt$ *and* $F = -\nabla V$.

Note that it is usually easy to reach the picture of classical mechanics from quantum mechanics by letting the quantum parameter $\hbar$ go to zero while it is usually unclear hoe to formulate the quantum mechanics from classical mechanics. So, this duality provide such an important guideline.

This duality tells us that we can extract the equation of motion in classical mechanics once we have the path integral formula for the fundamental solution of the Schrödinger equation (3.1). To get our generalized Newton's equation of motion (2.5) in spacetime, we shall follow the spirit of Geometric Approach (II) : first use this duality on the curved spaces (i.e., Riemannian manifolds) and then employ the Principle of covariance (c.f. [Do]) to obtain the corresponding equation in spacetime. In so doing we consider a family of time evolution equation :

$$\alpha\frac{\partial\Phi(x,t)}{\partial t} = \nabla^2\Phi(x,t) + W(x)\Phi(x,t) \tag{3.2}$$

whereα is a nonzero complex number with $Re(\alpha) \geq 0$. When α is a positive real number, this gives a diffusion (heat) equation and it is related to the theory of Brownian motion. When $\alpha = -\frac{2im_o}{\hbar}$ and $W = -\frac{2m_o}{\hbar^2}V$, this reduces to the Schrödinger equation (3.1).

The natural generalization of this family of time evolution equations on a Riemannian manifold M is given by

$$\alpha \frac{\partial \Phi(x,t)}{\partial t} = \triangle \Phi(x,t) + W(x)\Phi(x,t) \tag{3.3}$$

where$\triangle$ is the Laplace operator associated to the metric tensor g_{ij} of M.

According to the Classical-Quantum Duality, in order to catch the Newton's equation of motion on curved spaces M with metric tensor g_{ij}, we need to develop a path integral formulation of the fundamental solution of the Schrödinger equation.

Given any nonzero complex number α with $Re(\alpha) \geq 0$, we define the α-action, denoted by $\omega_\alpha(\gamma)$, for a curve $\gamma : [t_0, t_1] \to M$ in the Sobolev space $H^{1,2}([t_0, t_1], M)$, by

$$\omega_\alpha(\gamma) = \int_{t_0}^{t_1} -\frac{\alpha}{4} < \gamma^{'}(t), \gamma^{'}(t) > + \frac{1}{\alpha} W(\gamma(t)) + \frac{1}{6\alpha} R(\gamma(t)) dt \tag{3.4}$$

where $R(\gamma(t))$ denotes the scalar curvature of (M, g) at the point $\gamma(t)$. Note that the Sobolev embedding theorem implies that a curve γ in $H^{1,2}([t_0, t_1], M)$ is continuous since $\dim[t_0, t_1] = 1$ and $\omega_\alpha(\gamma)$ is well-defined.

When $\alpha = -\frac{2im_o}{\hbar}$ and $W = -\frac{2im_o}{\hbar^2} V$, (3.3) is the Schrödinger equation on curved spaces $(M, (g_{ij}))$ and (3.4) takes the form

$$\omega_{-\frac{2im_o}{\hbar}}(\gamma) = \frac{i}{h} \int_{t_0}^{t_1} \frac{m_o}{2} < \gamma^{'}(t), \gamma^{'}(t) > - V(\gamma(t)) + \frac{\hbar^2}{12 m_o} R(\gamma(t)) dt. \tag{3.5}$$

Next we shall discuss some suitable path subspaces of $H^{1,2}([t_0, t_1], M)$ that will be used to define the path integrals.

Definition (Path Spaces $\mathbb{D}_k(a, b)$). For $k \in \mathbb{N}$, $t_1 > t_0 \geqslant 0$ and x, $y \in M$ we define the path space $\mathbb{D}_k(x, t_0,\ y,\ t_1)$ to be the space of all broken geodesics $\gamma : [t_0, t_1] \to M$ from x to y with possible broken points at $y_j := \gamma(t_0 + \frac{j(t_1 - t_0)}{k})$, $j = 1, 2, ..., k-1$. Note that $x = y_0$ and $y = y_k$. Another way to describe γ is that a broken

geodesic passing the point y_j at time $t_0 + \frac{(j-1)(t_1-t_0)}{k}$. The curve $\gamma|_{[t_0+\frac{(j-1)(t_1-t_0)}{k},\ t_0+\frac{j(t_1-t_0)}{k}]}$ is a smooth minimal geodesic from y_{j-1} to y_j and thus it has the speed $||\gamma^{'}(s)|| = d(y_{j-1}, y_j)k/(t_1 - t_0)$ when s is in $(t_0 + \frac{(j-1)(t_1-t_0)}{k}, t_0 + \frac{j(t_1-t_0)}{k})$. From this description, we know that a broken geodesic γ in $\mathbb{D}_k(x, t_0,\ y,\ t_1)$ corresponds canonically to a ordered set of points $\{y_j\}_{j=1}^{k-1}$ when the point y_{j-1} is not in the cut locus of y_j (c.f. [BC]). As long as the integration is concerned, we can always assume this for all curves $\gamma \in \mathbb{D}_k(x, t_0,\ y,\ t_1)$. Under this assumption, $\mathbb{D}_k(x, t_0,$ $y,\ t_1)$ has a natural manifold structure of dimension $n(k-1)$ since it can be identified with the open submanifold of all points $(y_1, y_2, ..., y_{k-1})$ in M^{k-1} such that y_j is not a cut point of y_{j-1} for all $j = 1, 2, ..., k$. Note that in this way, $\mathbb{D}_k(x, t_0,\ y,\ t_1)$ can be viewed as an open submanifold of M^{k-1} and indeed, it is obtain ed by taking away a measure zero subset from M^{k-1}. We can also set the starting event point $a = (x, t_0)$ and the ending event point $b = (y, t_1)$ and denote the path spaces $\mathbb{D}_k(x, t_0,\ y,$ $t_1)$ briefly by $\mathbb{D}_k(a, b)$.

Thus, the path space $\mathbb{D}_k(a, b)$ can be endowed with a canonical measure, denoted by $\mathcal{D}_k\gamma$ from the product manifold M^{k-1} via the correspondence of γ and the ordered set $\{y_j\}_{j=1}^{k-1}$ viewed as the point $(y_1, y_2, ..., y_{k-1})$ in M^{k-1}.

Let $K_\alpha(b, a)$ denote the fundamental solution of the time evolution equation (3.3) at a. In [Wu1,2], we provide a mathematical theory of the path integral formulation for the family of time evolution equations (3.3). More specifically, we prove, under a suitable uniformality condition, the following

$$K_\alpha(a, b) \;=\; \lim_{k\to\infty} \int_{\mathbb{D}_k(a,b)} (\frac{k\alpha}{4\pi t})^{kn/2} e^{\omega_{\alpha_k}(\gamma)} \mathcal{D}_k\gamma$$

where $\omega_{\alpha_k}(\gamma)$ is given in (3.4) and $\alpha_k = \alpha + k^{-0.001}$ when $Re(\alpha)$ $= 0$, $\alpha_k = \alpha$ when $Re(\alpha) > 0$.

When α is a positive real number, for example 2, this generalized the Feynman-Kac formula to curved spaces and allows one to study Brownian motions on those spaces ([CW]). To our current interest, we take $\alpha = \frac{-2im_o}{\hbar}$ and $W = -\frac{2m_o}{\hbar^2}V$. Thus we obtain the fundamental solution $K(a, b)$ of the Schrödinger

equation on curved spaces : for $s = t_1 - t_0$

$$K(a,b) = \lim_{k\to\infty} \int_{\mathbb{D}_k(a,b)} (\frac{km_o}{2\pi\hbar si})^{\frac{kn}{2}} e^{\frac{i}{\hbar}\int_{t_0}^{t_1} \frac{m_o}{2} <\gamma^{'}(t),\gamma^{'}(t)> - V(\gamma(t)) + \frac{\hbar^2}{12m_o}R(\gamma(t))dt} \mathcal{D}_k\gamma.$$

The constant $(\frac{km_o}{2\pi\hbar si})^{\frac{kn}{2}}$ is the normalizing factor. This formula can be put into a physical notion as

$$K(a,b) = \int_a^b e^{\frac{i}{\hbar}S[\gamma(t)]}\mathcal{D}\gamma(t) \tag{3.6}$$

with the action

$$S[\gamma(t)] = \int_{t_0}^{t_1} \frac{m_o}{2} < \gamma^{'}(t), \gamma^{'}(t) > -V(\gamma(t)) + \frac{\hbar^2}{12m_o}R(\gamma(t))dt. \tag{3.7}$$

According to the Classical-Quantum Duality, the Euler-Lagrangian equation of the action $S[\gamma(t)]$ gives us the corresponding Newton's equation of motion:

$$F = m_o\nabla_{\gamma^{'}(\tau)}\gamma^{'}(\tau) - \frac{\hbar^2}{12m_o}\nabla R \tag{3.8}$$

with the external force $F = -\nabla V$.

So far, the equation (3.8) is a non-relativistic equation of motion in a curved space, but not in a spacetime. To go from Riemannian Geometry to General Relativity, we allow the metric tensor g_{ij} to be Lorenz and hence the principle of general covariance and the principle of minimal gravitational coupling (c.f. [Do]) allow us to obtain the desired equation of motion (2.5) with quantum-gravitation effect in spacetime. This completes our search for the equation (2.5) in (GR-II').

Next we shall use two different ways to derive the equation (2.4) in (GR-II'). First, we recall two of Einstein's results :

(1) Einstein's mass-energy formula : $E = m_oc^2$, and

(2) The energy E of massless particles, like photons, is proportional to its frequency ν : $E = h\nu$.

These two formulas together give us the mass of equivalence, still denoted by m_o for a massless particle :

$$m_o = \frac{h\nu}{c^2}. \tag{3.9}$$

To obtain the equation (2.4), we rewrite the equation (2.5) by using (3.9) and get

$$\frac{h\nu}{c^2}\nabla_{\gamma'(\tau)}\gamma^{'}(\tau) = \frac{\hbar^2 c^2}{12h\nu}\nabla R \tag{3.10}$$

when there is no external force.

Simplify (3.10) to yield

$$\nabla_{\gamma'(\tau)}\gamma^{'}(\tau) = \frac{c^4}{48\pi^2\nu^2}\nabla R \tag{3.11}$$

and this is equivalent to the equation (2.4) in (GR-II') since $\lambda\nu = c$.

Another way to obtain this equation is to use the notion of matter waves of Prince Louis de Broglie. According to de Broglie, a particle travelling with a certain momentum p has an associated matter of wavelength λ given by the relation:

$$\lambda = \frac{h}{p}. \tag{3.12}$$

Plugging the formula (3.12) into the equation (2.5), we get

$$\frac{h}{\lambda v}\nabla_{\gamma'(\tau)}\gamma^{'}(\tau) = \frac{\hbar^2 \lambda v}{12h}\nabla R \tag{3.13}$$

where v is the speed of the particle, when there is no external force.

Now for a massless particle, the equation (3.13) gives again the equation

$$\nabla_{\gamma'(\tau)}\gamma^{'}(\tau) = \frac{\lambda^2 c^2}{48\pi^2}\nabla R \tag{3.14}$$

and this completes our derivation of the intrinsic force, $\frac{\hbar^2}{12m_o}\nabla R$ and $\frac{\lambda^2 c^2}{48\pi^2}\nabla R$, due to the quantum-gravitation effect.

4 A new relativistic wave equation

In this section we shall introduce a new relativistic wave equation in spacetime by taking the notion of physical in spacetime into consideration. We also following the line of thinking of the Geometric Approach(II) as described in the introduction. Recall the Schrödinger equation on curved space takes the form:

$$i\hbar\frac{\partial\Phi(x,t)}{\partial t} = -\frac{\hbar^2}{2m_o}\triangle\Phi(x,t) + V(x)\Phi(x,t) \tag{4.1}$$

for a particle with mass m_o in a curved space. The equation is now included the quantum effect and the curved-space effect, but not the relativity effect. To unify these three effects, we take the curved spaces to be the spacetime. On one hand, the Riemannian notion of a 4-dimensional curved space becomes now a 4-dimensions Lorentz manifold (spacetime) as given by the Einstein's field equation. Thus a 3-dimensional position x_3 corresponds to a 4-dimensional spacetime position $x = (ct, x_3)$. The classical notion of time is now becoming the notion of proper time in general relativity. Thus correspondence can be put as

Position in a 3-dimensional space : x_3
$\longmapsto$ Position in a 4-dimensional curved space : x_4
$\longmapsto$ Position in a 4-dimensional spacetime : $x = (ct, x)$.

The correspondence of time for a physical event is

Absolute time in a 3-dimensional space
$\longmapsto$ Absolute time in a 4-dimensional curved space
$\longmapsto$ Proper time in a 4-dimensional spacetime

Under these correspondences, we can transfer the Schrödinger equation on curved space into a spacetime :

$$i\hbar\frac{\partial\Phi(x,\tau)}{\partial\tau} = -\frac{\hbar^2}{2m_o}\Box\Phi(x,\tau) + V(x)\Phi(x,\tau) + \omega\Phi(x,\tau) \tag{4.2}$$

where $\Box$ denotes the d'Alembertian operator associated to the Lorentz metric tensor g_{ij} and ω is a universal constant related to the relativity effect that need to be determined. The d'Alembertian operator in the Minkowski space is given by

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}.$$

To find the correct value of relativistic constant ω, we consider the special case that the Hamiltonian is conservative and thus this reduces to the equation

$$-\frac{\hbar^2}{2m_o}\Box\Phi(x,\tau) + V(x)\Phi(x) + \omega\Phi(x) = 0. \tag{4.3}$$

When the potential V vanishes, this equation should be equivalent to the classical Klein-Gordon equation:

$$\Box\Phi(x) + (\frac{m_o c}{\hbar})^2\Phi(x) = 0. \tag{4.4}$$

Thus the relativistic constant is $\omega = -\frac{m_o c^2}{2}$. Pluging this constant into the equation (4.2) we obtain
A relativistic wave equation.

$$i\hbar\frac{\partial\Phi(x,\tau)}{\partial\tau} = -\frac{\hbar^2}{2m_o}\Box\Phi(x,\tau)+V(x)\Phi(x,\tau)-\frac{m_o c}{2}\Phi(x,\tau). \tag{4.5}$$

In view of this equation and the generalized Newton's equation of motion (2.5), the correct action $S_{GR}(\gamma)$ for a path γ from [0,1] into the spacetime should take the form :

$$S_{GR}(\gamma) = \int_{t_0}^{t_1} \frac{m_o}{2} < \gamma'(t), \gamma'(t) > + \frac{\hbar^2}{12m_o}R(\gamma(s)) + \frac{m_o c^2}{2} ds. \tag{4.6}$$

In particular, in the Minkowski space a rest particle with mass m_o will have energy :

$$\begin{aligned} \frac{m_o}{2}&< \gamma'(s), \gamma'(s) > + \frac{\hbar^2}{12m_o}R(\gamma(s)) + \frac{m_o c^2}{2} \\ &= \frac{m_o}{2}(c^2 - 0) + 0 + \frac{m_o c^2}{2} \\ &= m_o c^2. \end{aligned}$$

This gives the well known formula : $E = m_o c^2$. In general, we have $E = m_0 c^2 + \frac{\hbar^2}{12m_0}R$ in a spacetime.

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